ELECTRONIC ANALOG MULTIPLIERS

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by

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PREFACE

The purpose of this paper will be to give the reader a general description of possible methods of multiplication by electrical analog systems and their possible advantages and failings. A report is given on the circuitry and performance of one type of multiplier, which, in the opinion of the author, best satisfies the requirements for a universal analog multiplier. As part of the curriculum in Engineering Electronics at the United States Naval Postgraduate School, the author spent ten weeks at Gilfillan Brothers, Inc., Los Angeles, California, working as an engineer in the computer group. It was here that the experimental work contained in this paper was performed.

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INTRODUCTION

Automatic computers find their use in automatic control systems and in the solution of scientific and engineering problems where the mathematical relations may be intricate or involved or the quantity of data to be handled is too large to be handled by available skilled personnel. There are two separate fundamental approaches to the problem of automatic computing system instrumentation, usually depending on the complexity commensurate with the accuracy desired.

Digital computers consist of counters registering and adding in discreet steps, together with a storage and programing system in which counting pulses are transmitted between counters in the manner prescribed by the problem to be solved. Digital computers perform most mathematical operations by combinations of additions, for example, multiplication is performed by repetitive additions, integration is by summation and converging series replace non-linear functions. Since these indirect computations must usually be carried to more places than are required in the final result, the computing elements must have the capacity for dealing with large numbers and thus contribute to the size of the installation. In automatic control systems, shaft position or electrical voltage information must be transformed to digital form before it is operated upon, then

reconverted to a usable voltage for use in the control system.

In analog computers the numerical values representing variables in the equation to be solved are converted to machine variables upon which the computing operations are performed. The machine variables may take the form of electrical voltages or shaft position, depending upon whether the system is in the form of a mechanical, electro-mechanical or strictly electrical system. In automatic control systems the inputs are more than likely already in the form of electrical voltages. Generally, more accurate results can be obtained from mechanical or electro-mechanical systems, but since the speed of computation in any mechanical system is limited by the inertia of its moving parts, this discussion will be limited to electrical systems.

It is desirable to make a computing machine as simple as possible. Accordingly, it is customary to perform more complicated mathematical operations on the computor voltages through combinations of a limited number of simple operations performed by basic computing elements. The necessary basic operations are as follows:

1) multiply a machine variable by a constant coefficient
2) take the sum or difference of two machine variables

3) generate the product of two machine variables

4) generate functions of a machine variable

5) generate the time integral or time derivative of a machine variable.

High gain direct coupled amplifiers with negative feed back make it possible to add, substract, multiply by a constant, integrate and differentiate with high accuracy and speed. But, the operations required in the generation of complicated functions and multiplication of variables are more difficult to perform, especially if both high accuracy and speed are desired.

Digital computers are inherently capable of much greater accuracies than analog computers, but these are obtained only at the cost of added instrumentation complexities; therefore, where the accuracy of the input data is limited and where great precision of computer operation is thus not required, the advisability of using an analog computer is indicated.

As stated above, there are two general classes of multiplication used in analog computers. The first is multiplication, in which one of the variables is constant throughout a given problem. This type does not present a very difficult problem, as there are many devices that can be hand set to obtain this result. An example is the high gain D.C. amplifier with controlled feedback. The second type is multiplication in which both of the quantities may vary in the

solution of a given problem. This is the type with which this report is concerned. This type of multiplier is needed in the solution of differential equations with variable coefficients. This type of equation is the more difficult to solve by classical mathematics, so it is important that not only automatic control systems but practical differential analyzers should include multipliers of this type.

Electronic analog computers call for multipliers with various requirements, depending upon the special applications. It would be desirable to have a universal multiplier which satisfies the most rigid of all the varied requirements.

It will be the aim of this paper to discuss examples of possible methods of electrical analog multiplication, while keeping in mind the requirements of a universal analog multiplier. After determining the method which best satisfies these requirements, one circuit which was built and tested by the author will be presented in detail.

II

POSSIBLE METHODS OF ELECTRICAL ANALOG MULTIPLICATION

1. Mathematical principles used in analog multiplication.

There are three important mathematical identities that have found use in analog multiplication. These identities permit the operations of addition and subtraction to be used instead of the more complicated operation of direct multiplication. This is a very useful substitution since the operations of addition and subtraction are easily performed in analog computers compared to direct multiplication.

1) Logarithms

$$\mathrm{Log}\,ab = \mathrm{Log}\,a + \mathrm{Log}\,b$$

This method is inherently restricted to positive numbers and requires the use of non-linear logarithmic elements.

2) Quarter square

$$ab = 1/4\left[(a+b)^2 - (a-b)^2\right]$$

This method requires the use of a non-linear square law element which will accept both positive and negative values.

3) Integration by parts

$$uv = \int udv + \int vda$$

This method is of use only in mechanical computers, since all integration in an electronic system must be performed with respect to time. The above expression requires integration

with respect to two other machine variables. Integration with respect to a machine variable in an analog computer can be accomplished by using the identity

$$\int x\,dy = \int (x\ dy/dt)\ dt$$

The procedure involves differentiation, multiplication, and then a final integration with respect to time. As can be seen, multiplication of two variables is required to perform the integration.

In addition to the mathematical identities mentioned above, several geometrical theorems are applicable for analog multiplication.

1) The area of a rectangle

   Area = a x b

   Where a and b are the lengths of two sides of a rectangle. In an electronic system one of the variables must be converted to time; this automatically restricts the multiplier to two quadrant operation. An averaging technique is then used to obtain the desired answer.

2) Altitude of a Right Triangle

   Altitude = (slope) x (base length)

   This method is very similar to the rectangular area

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method in that one machine variable must be converted to a time for base length. Instead of averaging to obtain the result, an accurate peak detector is required.

2. Analog multiplier requirements.

The discussion of existing methods of electrical analog multiplication which follows will be based on:

1) Complexity of circuitry involved. As is true in all electronic circuitry, increased complexity results in increased initial cost and usually greater maintenance problems.

2) Rapidity of solution.--In many differential analyzer applications the speed of solution is of minor importance, but in automatic control systems, and repetitive computers, solutions are often desired in a few milliseconds.

3) Accuracy of result.--The accuracy of multipliers is important since the overall accuracy of a computor system is generally limited to the accuracy of the associated multipliers. It follows then that the desired accuracy of multiplication is limited only by the accuracy of the input data. In physical systems, this is often limited to about one percent.

4) Polarity of acceptable input signals.--Often only one or possible two quadrant operation of a multiplier is required for a given problem. But, when

both plus and minus variables are expected for the inputs and the algebraic product is desired, a four quadrant multiplier is necessary. Switching methods for providing the required input polarities and placing the proper sign on the resulting product are possible, but become more complicated than existing four quadrant multipliers. Four quadrant operation can also be obtained by combining two simpler multipliers capable of two quadrant operation (1), but as in the above, the resulting combination of circuitry becomes excessively complicated.

5) Dynamic range of input variables and output product.-- The dynamic range of the multiplier is important since signals must over-ride any undesired noise in the system and often the theoretical absolute accuracy obtainable in a system is a function of the dynamic range.

3. Electro-mechanical methods of analog multiplication.

In electro-mechanical systems high accuracy commensurate with slower speed can be obtained by using Ohm's law in a variable conductance network. Since Ohm's law is a natural product, it seems quite logical that it should be used in an analog multiplier. The basic equation is:

$$E = IR$$

where E = voltage across the circuit

I = current in the circuit

R = resistance of the circuit

A common application of this principle is the potentiometer (2, 3, 4 and 5). A linear potentiometer is used for this purpose. One of the variables is the voltage impressed across the potentiometer, and the other variable is the position of the sliding contact. The voltage E_o is given by the equation $E_o = E_{in}\, \phi / \phi\ max$. Where ϕ max is the maximum possible angle of rotation.

This method requires that one of the variables be a mechanical position. If both of the variables are voltages, a servo system can be used to convert one of them to a mechanical position. The single potentiometer requires that one of the variables always be a positive quantity. This can be overcome by the use of a center tapped potentiometer with a push pull output.

The balanced bridge offers another method of obtaining a product (4). The bridge is kept balanced by a servo system which adjusts a rheostat in one of the legs of the bridge. The other three legs contain rheostats that are set by the variables of the problem. The condition for balance is $\frac{R_4}{R_2} = \frac{R_1}{R_3}$, or $R_4 = \frac{R_1 R_2}{R_3}$ where R_1, R_2 and R_3 are proportional to the input variables and R_4 is proportional to the desired

product. It is apparent that this method can be used for either multiplication or division, but is limited to one quadrant operation.

A variation of the balanced bridge utilizing low inertia strain gages has been developed to meet the requirements of .1 percent accuracy and an effective time constant of about 1 mil second (6). The variables to be multiplied appear as a-c voltages; one of the voltages, $X \sin w_c t$, controls, through an amplifier, the mechanical movement of a loudspeaker voice coil, whose movement produces fluctuating strains in a strain gage bridge (Fig. 1). This bridge is excited by a constant amplitude voltage, $V_o \sin w_c t$. The bridge output is fed back to the amplifier input in a negative sense. The mechanical displacement of the strain gage bridge is thus proportional to the variable voltage (X).

A second strain gage bridge is also coupled to the same loudspeaker voice coil so that its strain is also proportional to X. However, the second bridge is excited by the other variable voltage, $Y \sin w_c t$. Hence, the output of this second bridge is proportional to the product XY. But since the input variables are modulated a-c, the strain gage multiplier is a one quadrant device and requires considerable precision construction in order to obtain accurate results.

The dynamometer can be used with two electrical inputs to obtain a product as a mechanical rotation (7). The rotation

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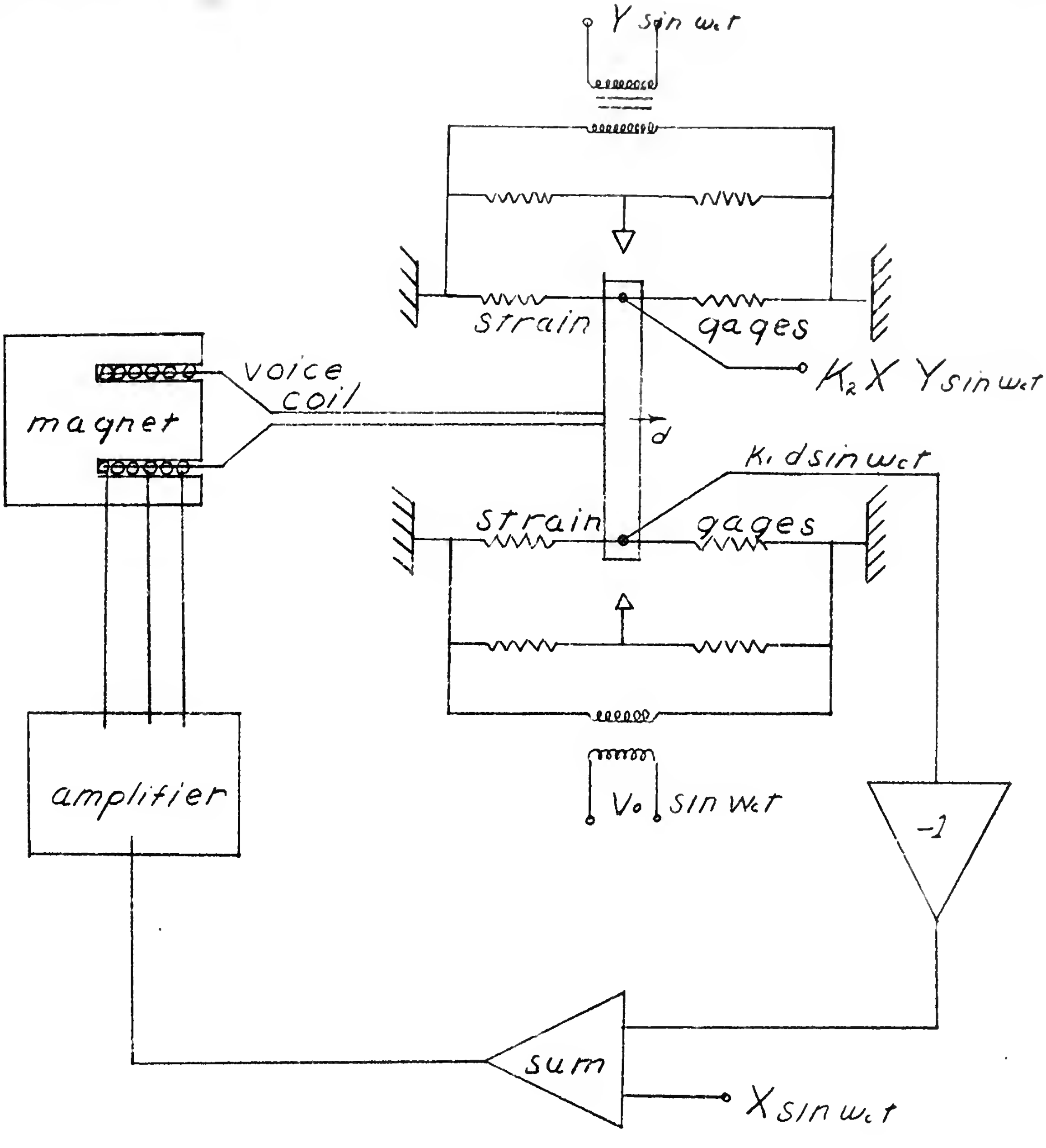

Strain Gage Bridge Multiplier

Figure 1

of the movable coil is given by the equation:

$$\emptyset = K \; Im \; Is$$

where $\emptyset$ = angle of coil rotation

K = proportionality constant

Im = current in movable coil

Is = current in stationary coil

This method is good for four quadrant operation and the angular output can be converted to an electrical voltage by a servo system.

A more refined, higher speed version of the dynamometer method utilizing two dynamometer movements rigidly connected on a common shaft is shown in (Fig. 2). (1) The torque from one movement is proportional to ($i_1 i_2$) and from the other movement $i_3 i_4$. When the two torques are equal and opposite, the rotational acceleration will be zero and the shaft will assume a position such that $K_1 i_1 i_2 = -K_2 i_3 i_4$. In other words, the sum of the shaft torques will be zero when $i_3 = -K \dfrac{i_1 i_2}{i_4}$. Therefore, if i_1 is made proportional to X and i_2 to Y and i_4 is a constant reference current, i_3 will represent the four quadrant product XY. The product $i_1 \; i_2$ causes a torque in the shaft resulting in a rotation of the mirror. As the mirror rotates from its center position, the voltages derived from the photocell outputs are compared, amplified and used to generate a

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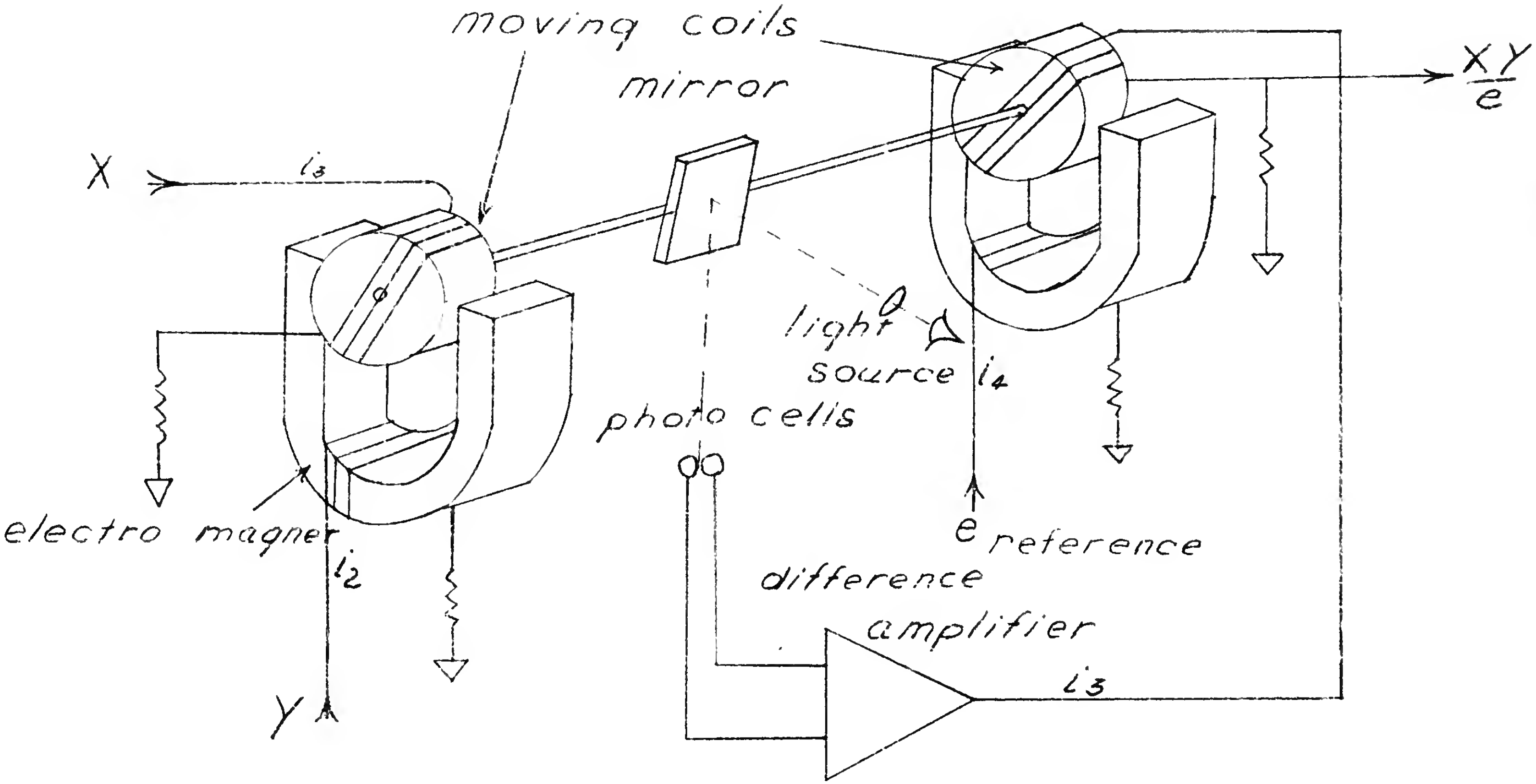
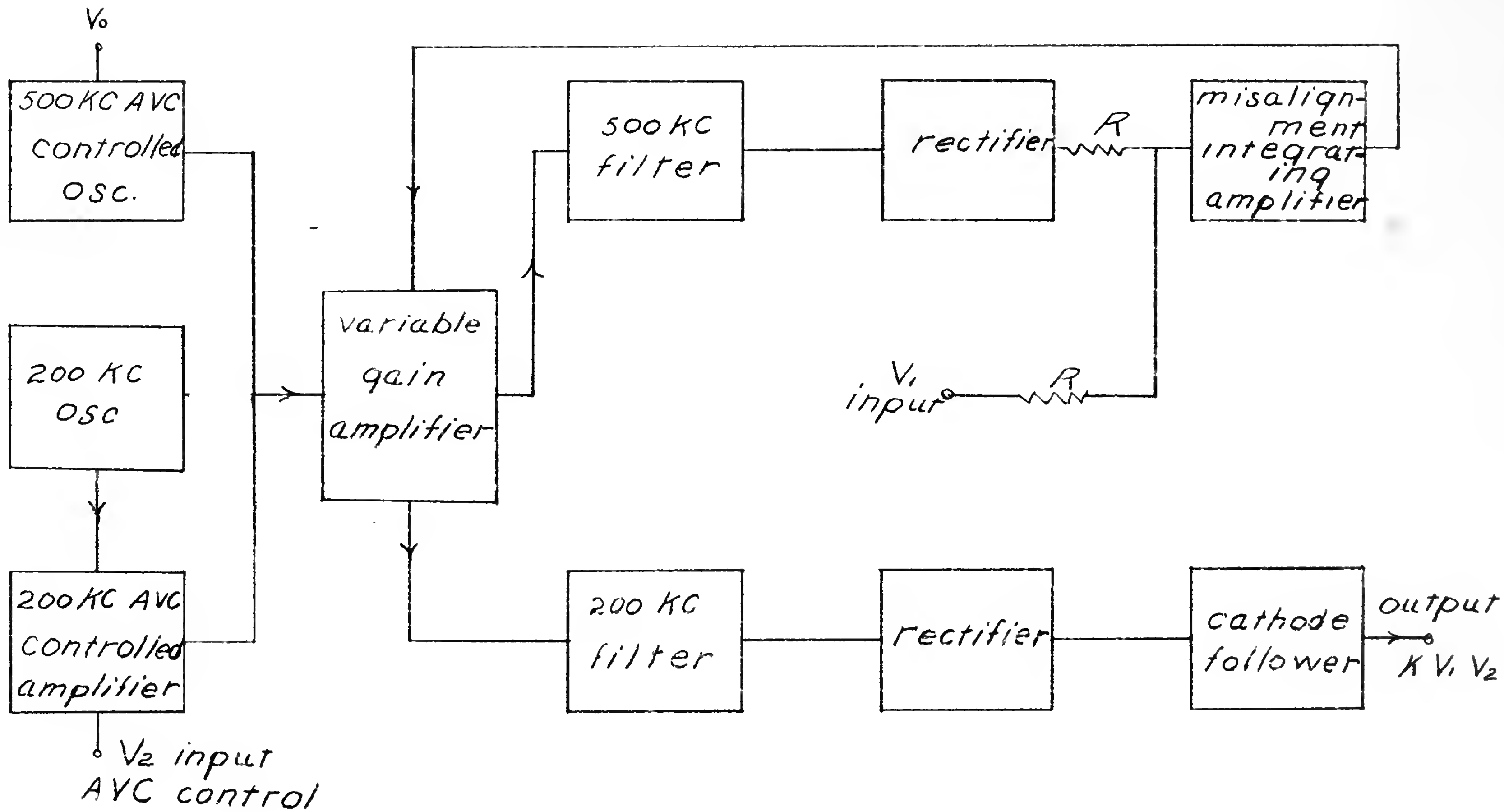


Two Dynamometer Multiplier
Figure 2

current, i_3, proportional to the difference voltage. The shaft will oscillate about its new position and settle down with $i_3 = -K \frac{i_1 i_2}{i_4}$. The speed of response is limited by the inertia of the moving parts and the inductance of the coils. Accuracy is effected by the linearity of the conversion from unknown voltage to current, the coulomb friction in the movement and the linearity between the coil flux and current.

4. Automatic gain control and modulation systems.

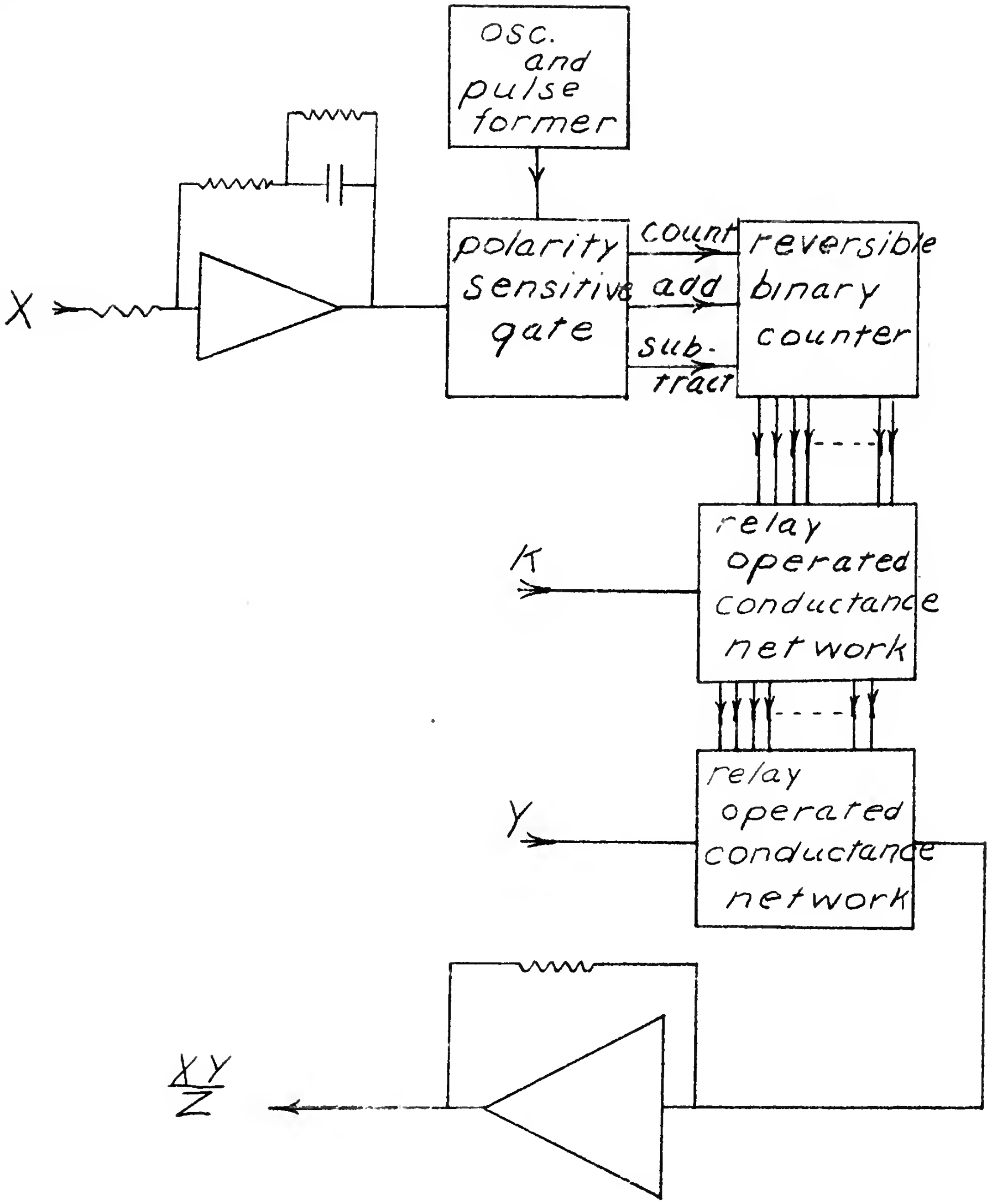
These systems may give accuracies of about 0.1 percent of the range of the output; that is, with a dynamic range of 100 volts, 0.1 percent accuracy can be obtained, and have a response time as low as ten micro-seconds. A typical example of an automatic gain control multiplier is shown in (Fig. 3) (8). A standard reference signal of 500 Kc is put through a variable gain amplifier. The 500 Kc component of the amplifier output signal is compared with one of the input signals, V_1, and the difference is fed back to control the amplifier gain. The result is that when the loop is in equilibrium, the gain of the amplifier is proprotional to V_1. A second signal, V_2, modulates a 200 Kc carrier, which is fed to the input grid of the amplifier. The output voltage at 200 Kc is then proportional to the product $V_1 V_2$. The chief advantage of this method of multiplication is that it does not depend (to the first approximation) on the tube characteristics and does not require unique components. The output of the system is only one quadrant.



Automatic Gain Control
Multiplier
Figure 3

Another example of a modulation system (9) utilizes a balanced modulator. An accuracy of better than one percent has been obtained; it uses a double modulation and subsequent detection scheme. A balanced modulator is used to produce side bands through modulation of a carrier by one of the variables. The resulting voltage is of the form E_1 cos (wct). This voltage is used as the carrier for a second modulation which gives a side band output of E_1E_2 cos (wct). The carrier voltage is suppressed in both cases through the use of a balanced modulator. Detection is then accomplished by a varistor bridge. This system is also limited to one quadrant operation.

A variation of the automatic gain control amplifier is the so-called step multiplier (10), which used relays to vary the input conductance by steps in a negative feed back amplifier. Although the system is not entirely electronic, the speed of response approaches that of an electronic system by the use of fast acting switching relays. The system makes the value of the input conductance proportional to one variable and the other variable is applied across the network. This is illustrated in (Fig. 4). The relays are set by a reversible binary counter. The counter is made to count pulses from an oscillator whenever the voltage fed back from the conductance network is different from the input variable voltage. The system was developed to obtain greater accuracy than is



Step Multiplier

Figure 4

possible with potentiometers. High speed, 100 micro-second, relays are used with an oscillator frequency of 1000 cps. There are 1024 steps in the conductance network, so that the output can go from minimum to maximum in about one second. The system accuracy is very good, but the one-second response must be considered slow.

5. Non-Linear systems.

The use of the quarter square identity and logarithms for analog multiplication was mentioned earlier. There are several applicable methods for obtaining the square law non linearity required. Among these methods are the use of specific tube characteristics, or non-linear materials. The transfer characteristics between grid voltage and plate current in a vacuum tube provides a somewhat approximate method for obtaining the square of a voltage. The instantaneous plate current of a triode with negative grid voltage expressed in a power series in terms of the grid exciting voltage e is:

$$ip = a_1e + a_2e^2 + a_3e^3 + \dots$$

For certain tubes, the plate current versus grid voltage characteristic is parabolic in form over a limited range of negative grid voltage values. Thus the voltage across the output resistence will be proportional to the square of the grid voltage. Suitable tubes must have a substantially constant Gm over the required range where

$$Gm = di_p/de$$

This results from the fact that the coefficient a_2 is given by

$$a_2 = 1/2 \, d^2e/di_p^2$$

A more precise parabolic transfer characteristic may be obtained by using two triodes in a balanced circuit designed to cancel odd power terms in the series expansion.

Circuits have been developed (11) in which the logarithmic relation between a low level applied voltage and the resistance of a rectifier, such as a selenium cell rectifier, is used to generate an output voltage proportional to the square of the input voltage. For low voltages, it has been found that when R is the rectifier resistance, e the applied voltage and K and q constants of the rectifier

$$R = K \, E^{-q\,e}$$

The current through the rectifier is therefore

$$i = \frac{e}{R} = \frac{e}{K} E^{qe}$$

The exponential may be expressed in series form as follows, neglecting quadratic and higher order terms on the assumption of small voltages:

$$E^{qe} = 1 + qe + \ldots\ldots$$

Hence $i = 1/K \, (e + qe^2)$

By subtracting the linear e/K term, a voltage proportional to e^2 will result.

Appropriate vacuum tubes can also be used to generate a logarithmic function directly. Variable mu tubes such as

This results from the fact [illegible]

by

$$[illegible] = 1/2 \, [illegible]$$

A more precise [illegible] transfer [illegible] may be obtained [illegible] to cancel [illegible] power [illegible] in the [illegible].

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voltage [illegible]

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The current [illegible]

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The exponential [illegible]

[illegible]

of [illegible]

...... + [illegible] + [illegible] = [illegible]

Hence [illegible] + [illegible] = [illegible]

[illegible]

[illegible]

the 6SK7 produce plate current proportional to the logarithm of the grid voltage. It has also been found (12) that by operating a 6SK7 in an inverted circuit, that is, with input voltage applied to the plate and current withdrawn at the grid, the output current is proportional to the antilog of the voltage at the plate.

A technique which uses biased diodes to switch appropriate conductances into a parallel circuit makes a non linear transfer characteristic by developing straight line approximations to a given curve. Circuits with accuracies as good as 0.4 percent and 12 micro-second response time have been built, using this method (13). The number of straight line sections used determines the closeness of the approximation to a desired curve. For example, a square law curve can be approximated to within two percent by three line sections and to within one percent by five line sections.

Another type of non linear function generator, called the photoformer (14, 15), is essentially an electronic servo system which makes a cathode ray beam follow the edge of an appropriate mask (cut to represent the desired function). A simplified block diagram is shown in (Fig. 5). The horizontal deflection represents the input variable and the vertical deflection, the output variable. Here the input variable is received from a signal source and applied to the horizontal deflection plates so that the spot is positioned on the 6SK7 produce [illegible] current proportional to [illegible] of the grid voltage. It has [illegible] operating a 6.7 in a [illegible] circuit, [illegible] voltage applied to the plate and [illegible] the output current is proportional [illegible] tage at the plate.

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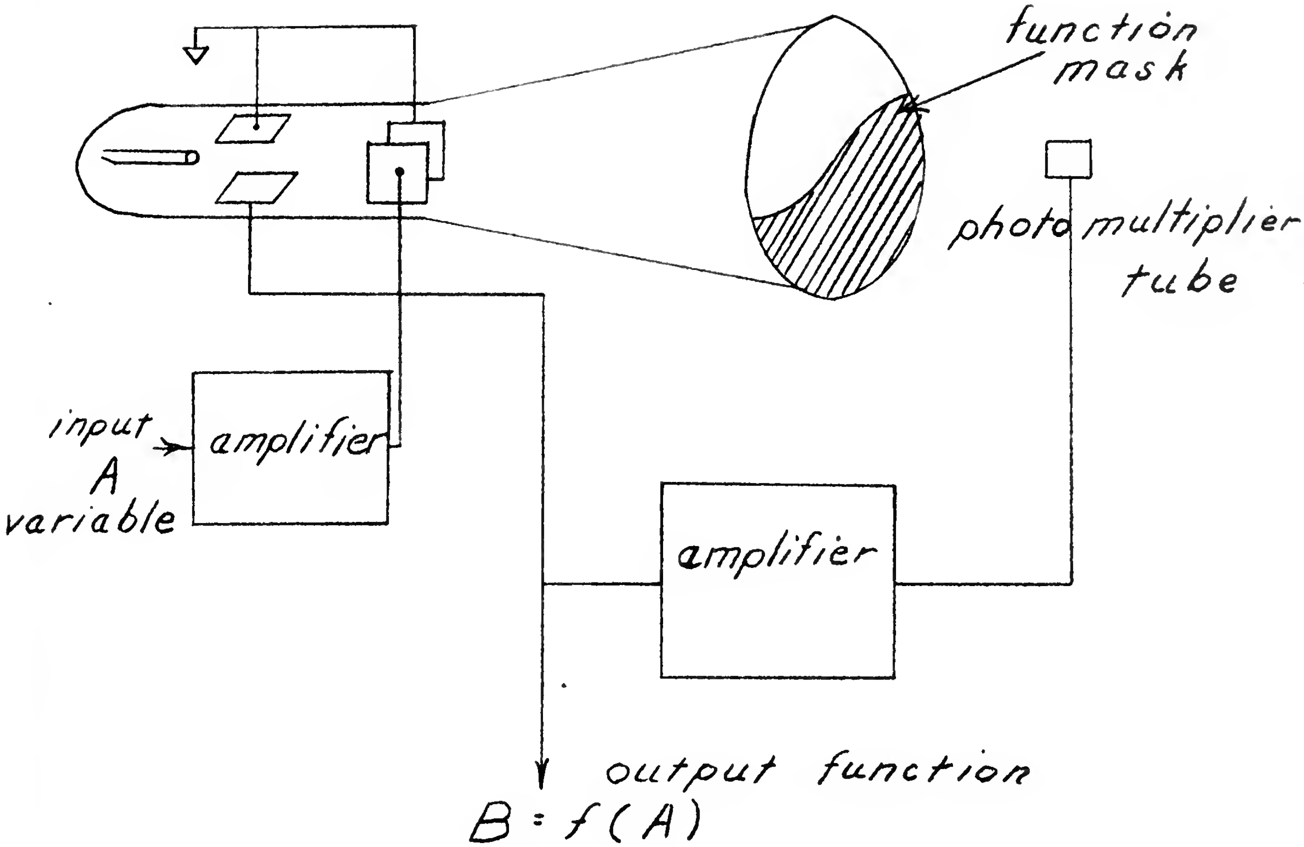


Simplified Photoformer Block Diagram

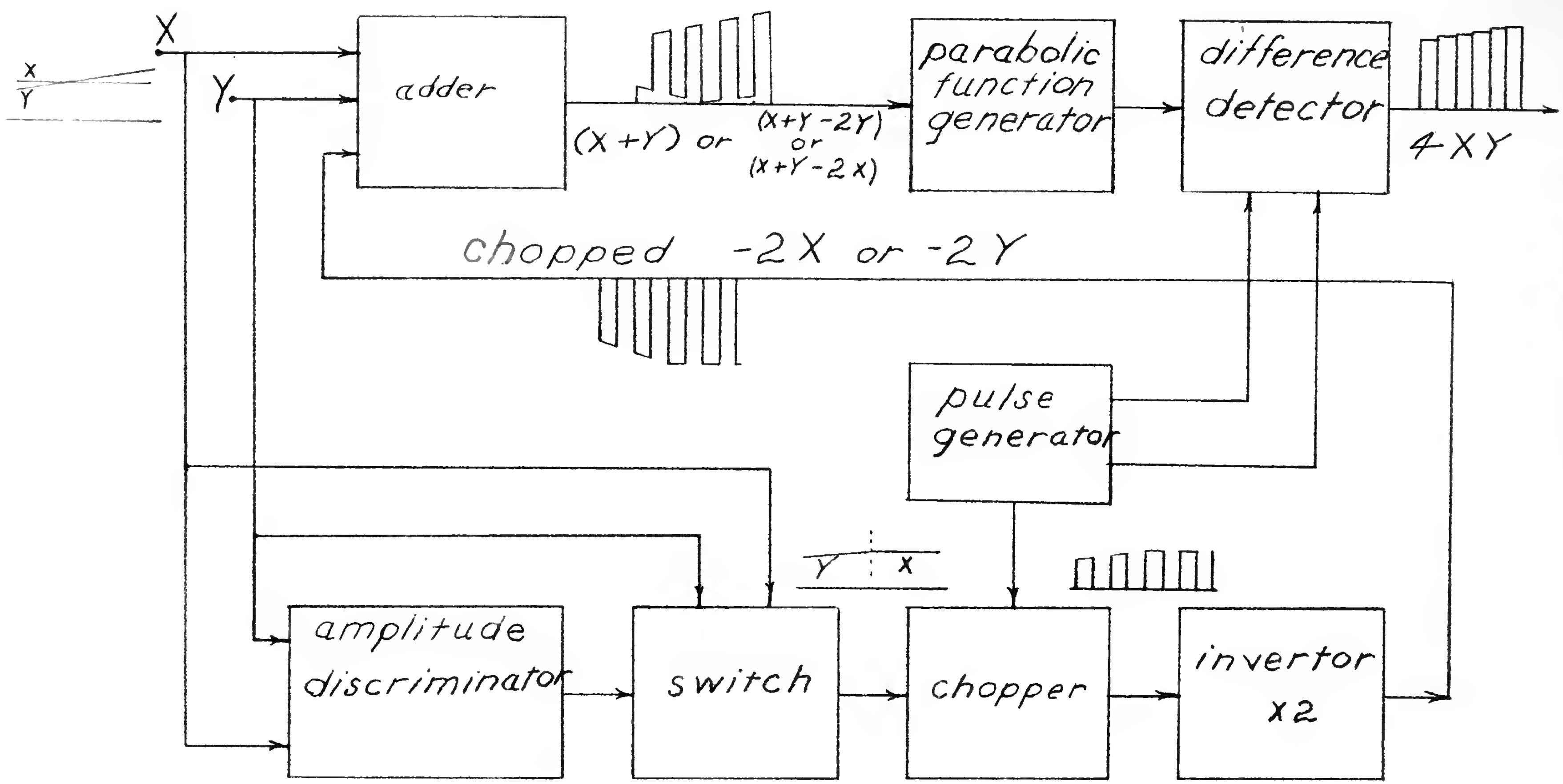
Figure 5

the edge of the mask. The precision is limited by the sharpness of the beam focus, as the spot cannot accurately follow details which are smaller than the spot itself. By using a lens system to improve the focusing and reduce parallax, accuracies better than one percent have been obtained. This accuracy may be limited by the signal to noise ratio in the feed back path.

Cathode ray tubes especially designed for producing a single transfer function have been developed, which operate essentially like the photoformer, but the external mask and photocell are replaced by a target and collector electrode inside the cathode ray tube envelope. One such a device, called the monoformer (16), employs a standard cathode ray tube base. The target is an aluminum disc of one inch diameter, on which the function to be reproduced is printed with a coating of carbon ink. The operation of the tube depends on the fact that aluminum and carbon have different secondary emission ratios. An electrode is provided to collect the secondary electrons emitted from the target plate as a result of impingement by the electron stream. If the beam, in sweeping across the target, tends to ride too far into the uncoated area or into the coated area, it causes a variation in secondary electron emission. The corresponding variation in target current produces a variation in voltage drop across a load. This error signal is fed back

through a network to the vertical deflection plates of the tube. The result is to keep the electron beam directed against the boundary between the coated and uncoated areas of the target. The accuracy of the unit is considered to be one percent without amplification in the feed back loop. A response time of 400 micro-seconds to a step input is developed. With sufficient amplification in the feed back loop, the response time reduces to one micro-second.

The above methods of generating non-linear functions can be used in multipliers based on the logarithmic or quarter square identities. A block diagram of a quarter square multiplier is shown in (Fig. 6). An error of less than one percent of maximum operating range has been obtained, with a solution time of about 50 micro-seconds (13). In this circuit, only one squarer is used on a time sharing basis. When two function generators are used, they must be very nearly identical. The error due to a small discrepancy in the squarer characteristics will be considerable if one of the inputs is large and the other small, because here the difference of the square of two large quantities is involved. The parabolic function generator used is of the biased diode network type. In the system, the sum term is generated by adding the X and Y input variables. But since the input to the squarer must be positive, a method is provided to insure that the difference term is never less than zero. The



Quarter Square Multiplier Using
Segmented Parabolic Characteristic

Figure 6

X and Y inputs are compared in an amplitude discriminator and fed to an electronic switch which passes only the smaller of the two inputs. The smaller of the variables is then chopped into equal on and off pulses, multiplied by two and subtracted from the sum term. Thus, the input to the parabolic function generator alternates between the sum and difference terms, but is always positive. The difference between the amplitudes of the adjacent squared sum and difference pulses is determined in a difference detector and is equal to the output, 4 X Y. The circuit can handle only positive input variables and has a dynamic range of 0 to 25 volts. The circuit is rather complex, with five d c amplifiers, twenty-six diodes and fifteen other tubes of various types, exclusive of the squarer and pulse generator.

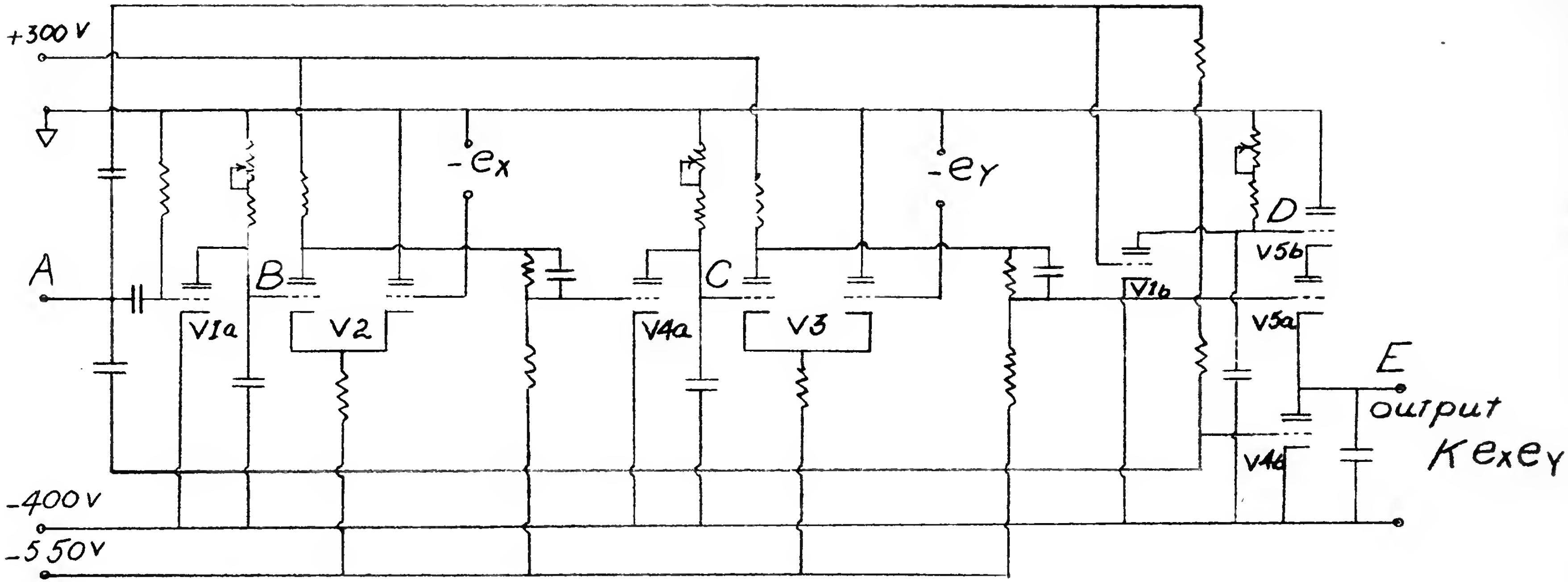
Exponentials can also be used for analog multiplication. A circuit based on the following mathematical expressions gives promising results:

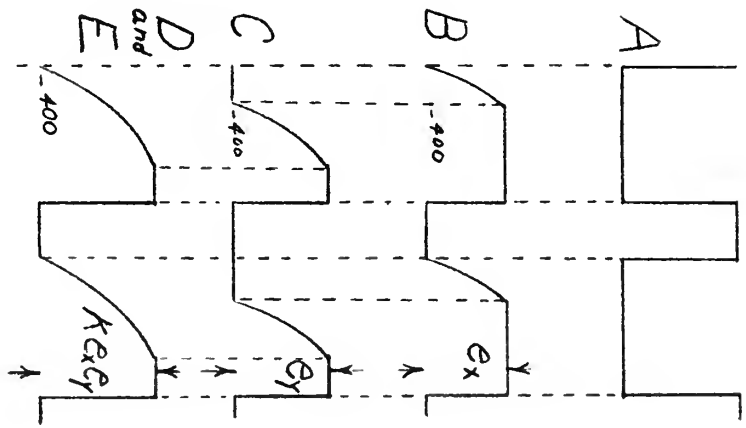
$$\text{if } V_1 = -A\ e^{\frac{-tx}{T}}$$

$$\text{and } V_2 = -A\ e^{\frac{-ty}{T}}$$

$$\text{then } V_1V_2 = A^2\ e^{-\frac{(tx + ty)}{T}}$$

The circuit shown in (Fig. 7) and described below will give the response $-A\ e^{-\frac{(tx + ty)}{T}}$, thus differing from its desired answer by a scale factor which can easily be taken into account.





Exponential Multiplier

Figure 7

Figure 7 shows the schematic diagram and the idealized wave forms present at various strategic points in the circuit. At point A, a square wave with a one millisecond period is fed to the grid of V_1a, which is a normally fully conducting tube. Point B normally resides in the neighborhood of -400 volts. When, however, tube V_1 is driven beyond cut-off by the negative half cycle of the square wave input, point B will proceed to rise exponentially towards ground potential. The duration of its rise will depend on the period of the negative half cycle and the rate at which it will rise will depend on the RC time constant in the plate circuit of tube V_1a. The input variable voltage, ex, will set the cathode potential of tube V2 by virtue of the cathode follower action. When the potential at the grid of V2, point B, reaches the grid conduction region, it will be prevented from further rising by the low effective grid to cathode resistance. The wave form will be as shown in (Fig. 7B). This signal will be amplified and inverted by tube V2 and applied to the control grid of V4A. A similar action will take place here, except that it will be initiated at a later time. The resulting signal will be amplified and inverted by tube V3 and applied to the control grid of tube V5a. Also, at the beginning of the entire sequence, the square wave input is applied to the control grid of tubes V1b and V4b. The negative half cycle of the square wave will initiate an exponential rise at points D

and E, while the signal to the control grid of tube V5a from tube V3 will end it. The signal to tube V4b is to discharge the storage capacitor between reading times. Results obtained from this circuit to the present time have not been conclusive, but it is felt that further refinements in the design could make this a good and simple one quadrant multiplier.

6. Cathode ray tube systems.

The crossed field electron beam multiplier (17) uses an electro-static deflection cathode ray tube in conjunction with a feed back amplifier and a photo multiplier tube. The electron gun of the cathode ray tube generates a sharply focused beam of electrons. The force on a stream of electrons moving with average velocity, v, at right angles to a magnetic field H is counteracted by an electro-static field E proportional to the product vH. If the adjustment of E were automatic and instantaneous, then its value would be a continuous measure of this product. In practice, v is proportional to a voltage V_x applied to the horizontal deflection plates of the cathode ray tube; H is proportional to a current I through a coil wound around the vertical deflecting plates of the cathode ray tube; and E is automatically adjusted by means of a mask, phototube and amplifier (as in the photoformer discussed above). Then E will be proportional to the product IV_x. This system gives four quadrant

operation, an absolute indication of zero (freedom from zero drift) and is independent of normal changes in electrical characteristics. As the important parameters are geometric, the prospects of high stability are good. The inductance of the magnetic field coil is the main factor which limits the speed of response of this device. The solution time is about 500 micro seconds with an accuracy of two percent.

Another principle of multiplying by cathode ray tubes makes use of a square beam of electrons which is deflected horizontally by the X input voltage and vertically by the Y input (18). The deflection causes the beam to fall eccentrically on four square collector plates (Fig. 8). The current from each plate passes through a load resistance. If the beam current density at impingement is uniform over the area 2L x 2L, the current through each load resistance is proportional to the area of impingement on the corresponding collector plate, which in turn is a function of both the X and the Y voltages. These areas are:

$$(L-x)\ (L+y) = L^2 + (y-x)\ L - xy \qquad 1$$

$$(L-y)\ (L-x) = L^2 - (y+x)\ L + xy \qquad 2$$

$$(L+x)\ (L+y) = L^2 + (y+x)\ L + xy \qquad 3$$

$$(L+x)\ (L-y) = L^2 - (y-x)\ L - xy \qquad 4$$

If equations two and three are added, one and four subtracted, the net current is found to be 4 x y, or the same as the result of the quarter square multiplier. Although good accuracy association, an [illegible] deviation [illegible]

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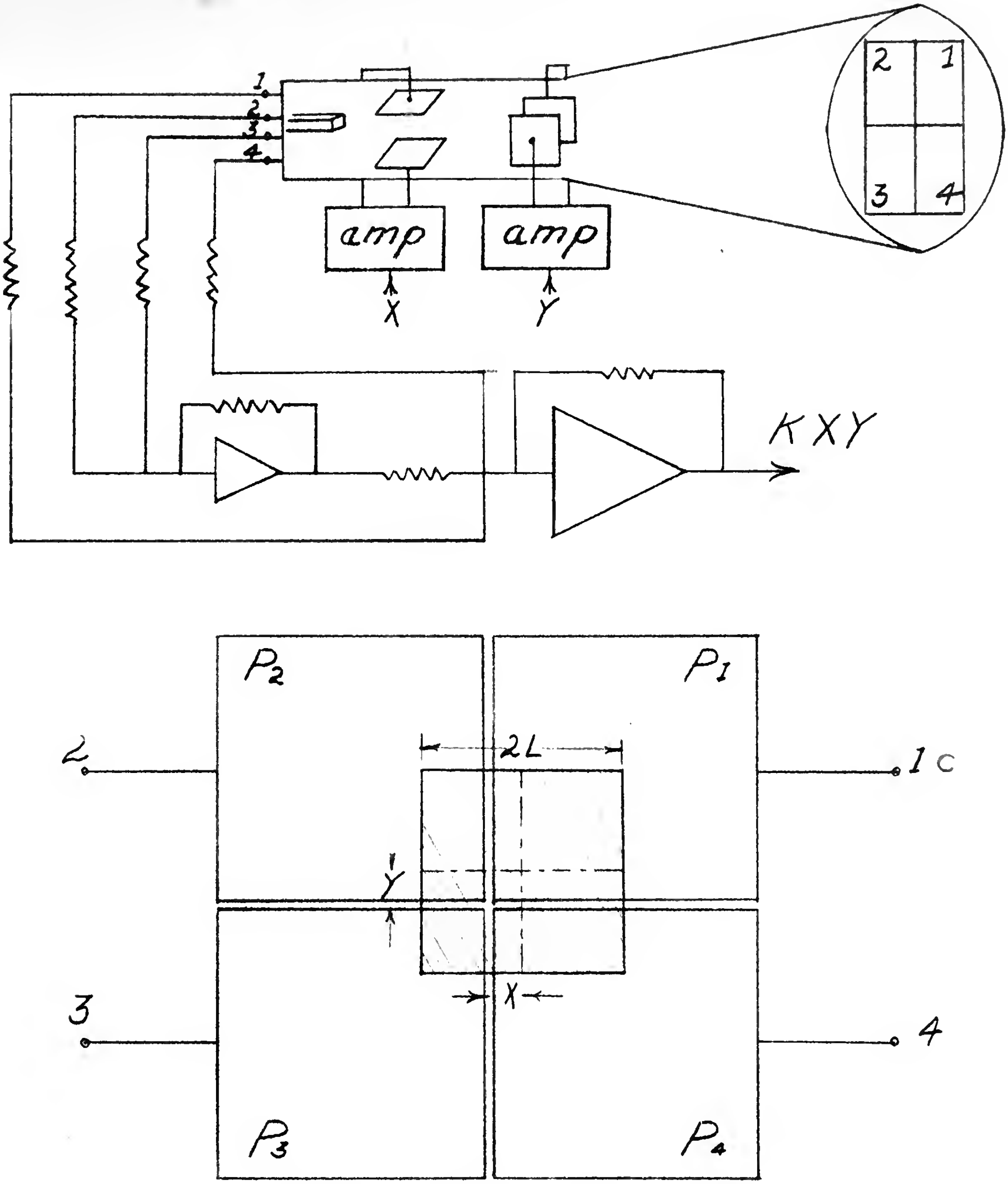


Square Beam Cathode Ray Tube Multiplier

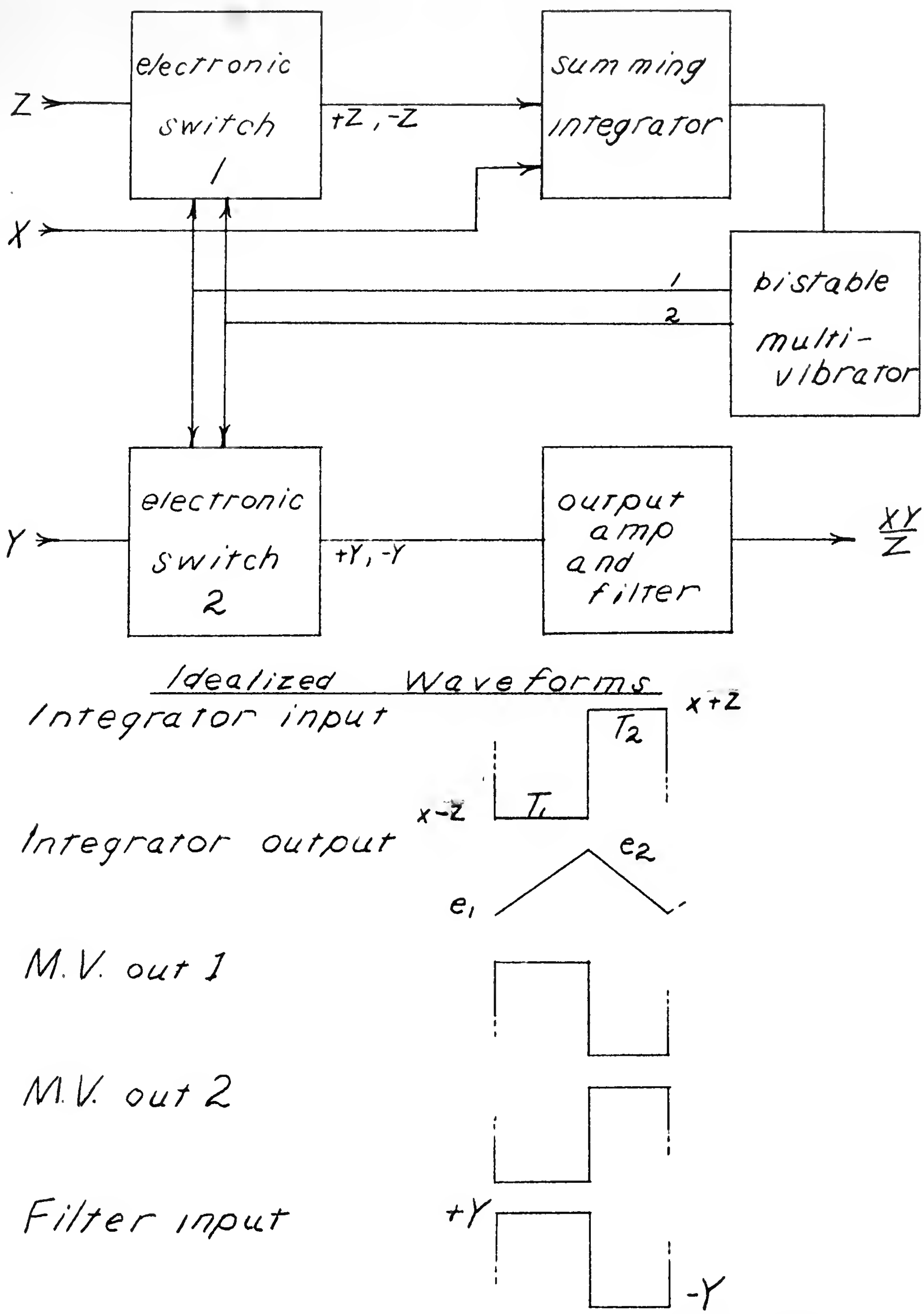
Figure 8

was not obtained with the first experimental tubes, the speed of response is not limited by an inductive circuit, as in the crossed field multiplier.

7. Time division multiplier systems.

Pulsed attenuator multipliers using a combination of pulse width and pulse amplitude modulation have been used successfully (3). They use the area of a rectangle technique mentioned earlier, where the average value of a pulsed voltage is equal to the product of the pulse amplitude, and the pulse width averaged over a full cycle. The pulse amplitude is made proportional to one input variable and the ratio of the on and off time of the pulses is proportional to the other input variable. This type of multiplier is limited to one quadrant operation and the switching tubes must be carefully matched in order to obtain the high accuracy of 0.2 percent, which has been obtained.

An extension of the pulsed attenuator technique called time division (19, 20) makes four quadrant operation possible and eliminates the necessity for carefully matched tubes. The basic principle of operation is that the algebraic product of two variables is formed by averaging several cycles of a quasi rectangular wave form. The duration and amplitude of the wave form are functions of the input variables, as shown in (Fig. 9). The amplitude of the portion T_1 is $+Y$, where $T_1 = \frac{K}{Z-X}$ seconds, and the amplitude of the portion T_2 is $-Y$,



Time Division Multiplier Block Diagram

Figure 9

where $T_2 = \frac{K}{Z+X}$ seconds. The average value of the complete cycle is $\frac{Y\,(T1 - T2)}{T1 + T2}$ or $\frac{XY}{Z}$. The basic wave form is produced by the switching process shown in (Fig. 9) and described below.

The pulse timing is dependent on input variables X and Z. It is controlled by a closed loop feedback system consisting of switch 1, the integrator and bistable multivibrator as shown in (Fig. 9). The multivibrator changes from one of its stable states to the other whenever the output of the integrator reaches e_1 or e_2 and actuates switch 1 and switch 2 simultaneously. For simplicity of explanation, the period when switch 1 is open will be called T_1 and when it is closed T_2. The output of switch 1 during T_1 is $-Z$, and during T_2 is $+Z$. Therefore, the input current of the summing integrator during T_1 is $x - z$. Switch 1 remains open until the output of the integrator reaches e_2, at which time the multivibrator changes states, closing switch 1 and changing the integrator input current to $x + z$. The output of the integrator then changes from e_2 to e_1 and continues to repeat the switching cycle. The transition time T_1 is computed from the integrator response to a step function, as follows:

Let $(e_1 - e_2)$ = voltage excursion required at input of bistable multivibrator to change its state

C = capacity of integrating condenser in farads

R = input resistor for variables in ohms

$x = \frac{X}{R_x}$ = input current to integrator due to variable X

$z = \frac{Z}{R_z}$ = input current to integrator due to Z.

Then: $e_1 - e^2 = -1/C \int_0^{T_1} (x-z)\, dt$

This equation assumes a high gain amplifier and an integrating capacitor with a high leakage resistance. Then, assuming that X and Z are constant during the period:

$$T_1 = \frac{C(e_1 - e_2)}{Z - X}$$

and similarily:

$$T_2 = \frac{C(e_1 - e_2)}{Z + X}$$

The variable Y is switched through the use of switch 2, which is actuated by the same switching pulses as switch 1. The rectangular wave form of (Fig. 9) appears at the output of the final amplifier if filtering is not performed. The average of this wave form, E_o, may be computed as follows:

$$E_o = \frac{+Y T_1 - Y T_2}{T_1 + T_2}$$

$$E_o = \frac{+\frac{YC(e_2 - e_1)}{Z - X} - \frac{YC(e_2 - e_1)}{Z + X}}{\frac{2ZC(e_2 - e_1)}{Z^2 - X^2}}$$

$$E_o = \frac{XY}{Z} = K\frac{XY}{Z}$$

Thus the average value of the output voltage is proportional to the product of the two variables, X and Y. Since d-c amplifiers with the required dynamic range and stability are available, the accuracy, response time, and dynamic range of the multiplier are dependent upon the type of electronic switch used. Circuits of this type with accuracies within 0.1 percent have been built.

8. Choice of system for construction.

It is evident from the discussion of electrical analog multipliers that the requirements of a universal multiplying device are difficult to fulfill. In order to satisfy all of the requirements, the multiplier must be a four quadrant device which combines a short response time with high accuracy over a large dynamic range. The requirement of high speed limits the selection to an all electronic device. The requirement for a four quadrant device limits the selection even further, unless added complex circuitry is acceptable to convert a one or two quadrant multiplier to four. The only electronic analog multipliers which are fundamentally four quadrant devices are the crossed field cathode ray tube

multiplier, the square beam multiplier, the quarter square multiplier with a photoformer squarer and the time division multiplier. To the author's knowledge, the square beam multiplier is not an accurate device in its present stage of development and other cathode ray types tend to be bulky and complex.

Through a process of elimination, the time division multiplier is the only type which approaches the requirements set down for a universal multiplier. A circuit of this type was constructed and tested by the author and will be discussed in more detail in the following chapter.

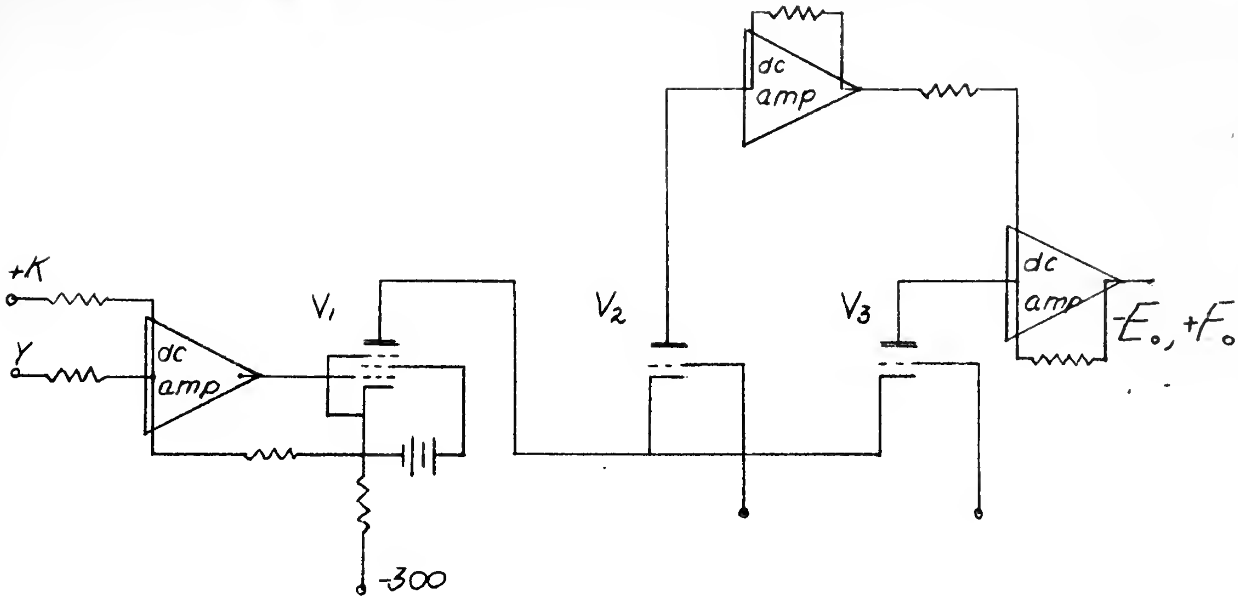
III

A TIME DIVISION MULTIPLIER

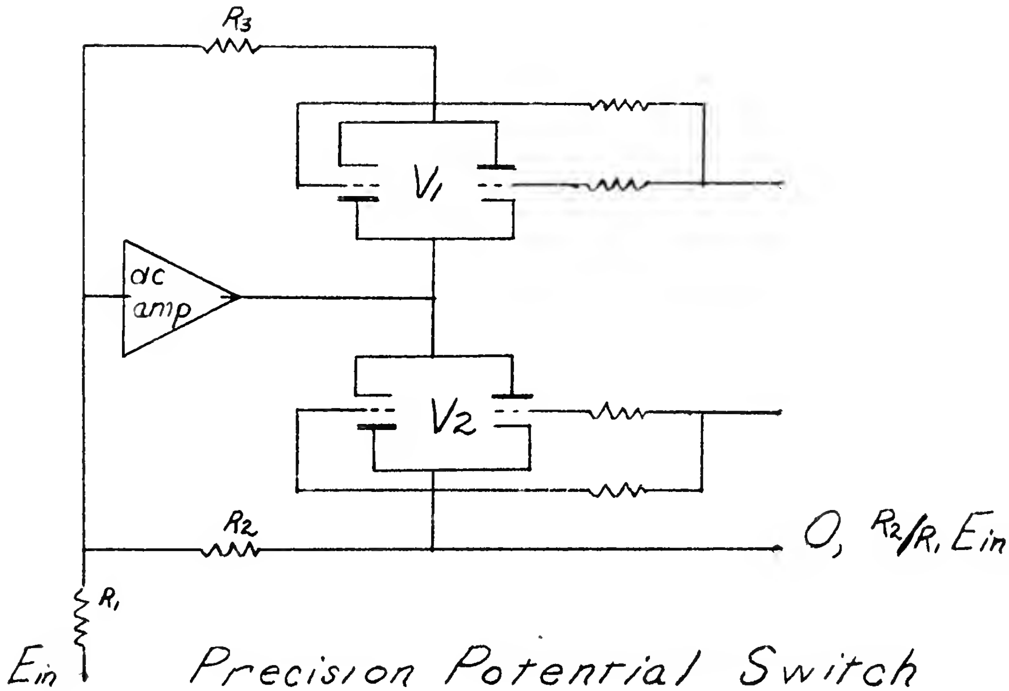
1. Precision switch.

The basic principle of operation of the time division type of multiplier was stated in the previous chapter. It is obvious that high accuracy and proper pulse timing is dependent upon an electronic switch with excellent high speed, precision characteristics. The switch and its associated circuitry must have the following characteristics: when the switch is in one condition, the current or voltage out must be a linear function of the voltage input; when in the alternate condition, the current or voltage out must be the negative of that in the first condition; in addition, the characteristics of the switch must be independent from normal variations in the tubes employed, must have a large dynamic range, a high input impedance and a low output impedance. Two switches suitable for the application will be discussed. The first is a so-called current switch, rather than a potential switch.

(Fig. 10) is a diagram of the electronic current switch (20). Switching signal voltage levels are applied to the grids of the triodes V_2 and V_3. If the control grid of V_2 is positive with respect to the control grid of V_3 by a sufficient amount, the plate current of the pentode, V_1, will flow through tube V_2 and the output voltage, E_o, will



Precision Current Switch
Figure 10



Precision Potential Switch
Figure 11

have a negative polarity. If the control grid of V_3 is positive with respect to the control grid of V_2 by a sufficient amount, the plate current of V_1 will flow through V_3 rather than through V_2, and E_o will have the same magnitude as before, but it will have a positive rather than a negative polarity. The plate current of V_1 can thus be made to flow in either of two external circuits. Since the switch is unidirectional and the range of operation is limited, it is necessary to add a fixed voltage to the variable voltage at the input. The unwanted component derived from the fixed voltage is eliminated from the output of the multiplier by a bridging system.

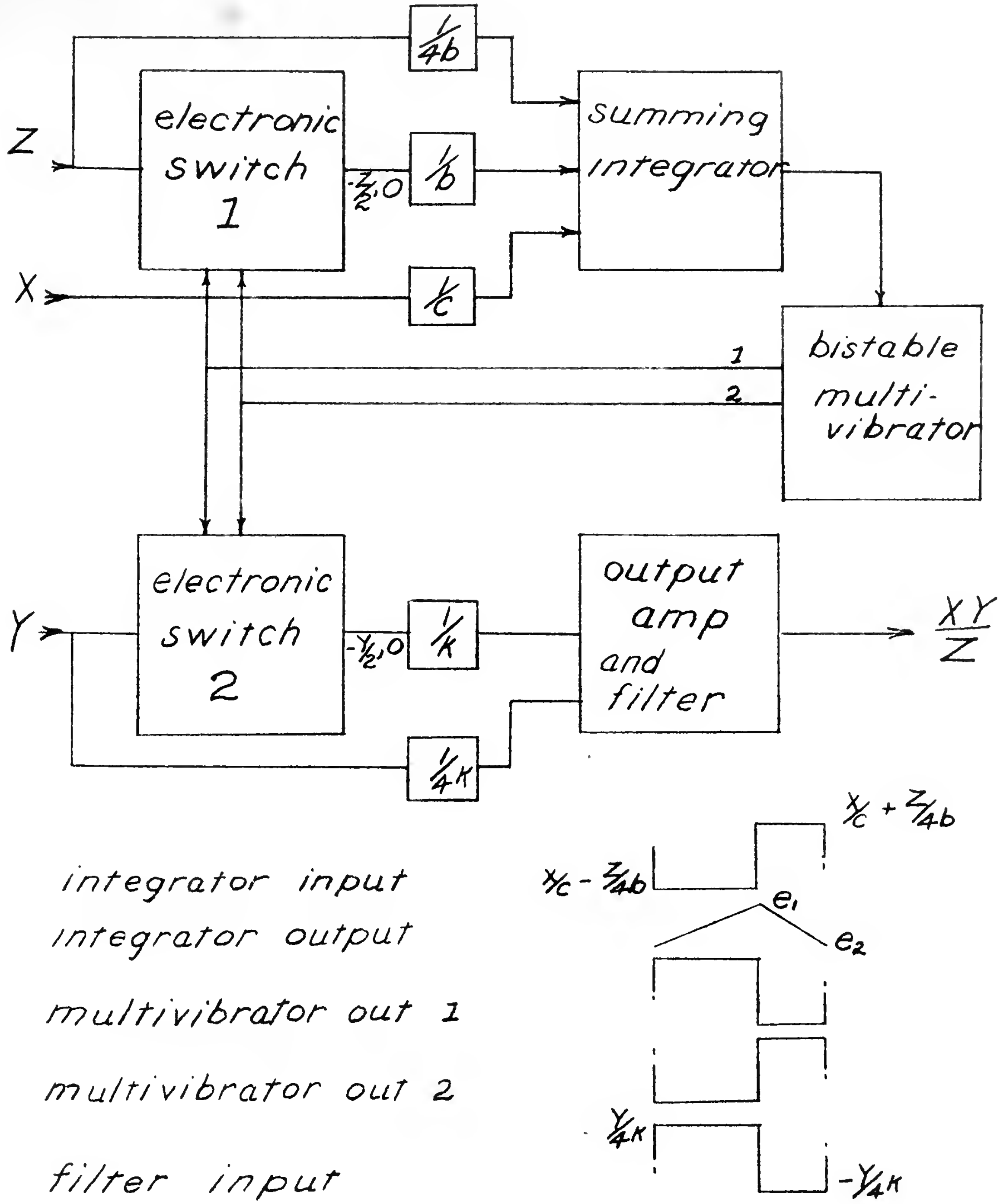
(Fig. 11) is a diagram of an electronic potential switch with suitable characteristics (19). It consists of a d-c amplifier with two alternately switched feed back impedances. The switch tubes in series with the two feed back impedances are connected so that when one is conducting, the other is cut off. The output voltage is taken from the junction of V_2 and R_2. When V_1 is on and V_2 is off, the output voltage is zero, since the junction of R_1, R_2 and R_3 is maintained at ground potential by the high gain negative feed back amplifier. When V_1 is off and V_2 is on, the output voltage is equal to $- R_2/R_1$ Ein. When a linear function of the voltage input equal to $1/2\ R_2 / R_1$ Ein is added to the switch output, a rectangular wave form symetrical about the zero axis is obtained.

Although both switches would perform satisfactorily in the circuit, the potential switch was selected since it is generally less elaborate than the current switch. It requires one less d-c amplifier and no battery bias supply, and does not produce an unwanted component in the output.

The results of tests indicated that the desired high speed, accurate switching could be accomplished with this rather simple circuitry, if the effects of distributed capacitance were kept to a minimum. This indicates that the feed back resistances must be kept small and the amplifier gain high. A compromise had to be made in determing the size of the resistance, however, since too small a feed back impedance reduces the dynamic range of the amplifier.

2. Circuit design.

A block diagram of the multiplier, incorporating the potential switch, is shown in (Fig. 12). The size of the ratio resistances, b, c, and k are established as a result of several considerations. Zmin/4b must be greater than Xmax/c, since the output of the integrator must reverse direction. Larger resistances will reduce the loading effect on the previous stage, but since the integrator is based on switching wave forms with short rise times, the effects of distributed capacitance will be minimized by using small resistances. The lengths of the times T_1 and T_2 are also affected by the magnitude of the ratio resistances.



Time Division Multiplier
Block Diagram

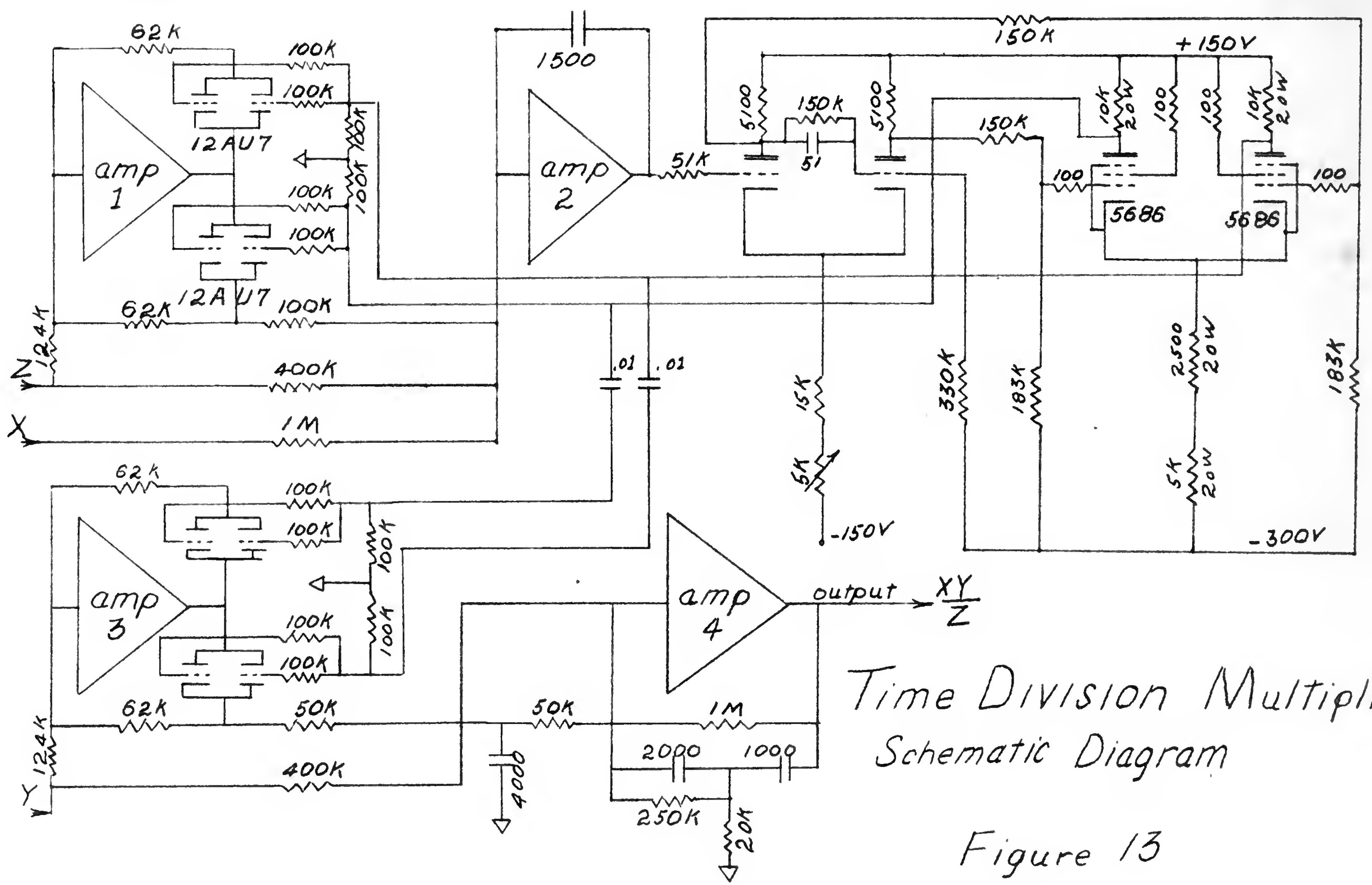
Figure 12

A schematic diagram of the multiplier circuit is shown in (Fig. 13), where the triangles represent high gain d-c amplifiers. The amplifiers have a differential input, a stage with regenerative feed back and a cathode follower output. The decision to use amplifiers without automatic balancing was prompted by the desire to make the circuitry as simple as possible. The dynamic range of the multiplier is limited by the amplifiers used and, in this case, is from minus fifty to plus fifty volts.

Certain scale reduction is necessary in an electronic multiplying circuit, in order to reduce the maximum swing of the output to the dynamic range limit of the final amplifier. The scale factor of the final amplifier here is made equal to kc/b, so that the multiplier will give the output XY/Z. It is evident from the relationship for the output voltage that the circuit can also be used for division by the variable Z. However, when Z varies, the frequency of the switching wave form varies widely and increases the difficulty of filtering the output. When the circuit is to be used only for multiplication, the variable Z should be set at a constant reference value, thereby establishing the desired scale factor for the final amplifier.

3. Output filtering.

If the input variables to the multiplier are varying d-c voltage, or very low a-c, (that is, if the required



Time Division Multiplier
Schematic Diagram

Figure 13

frequency response is very low), a simple high pass feed back network around the final amplifier is all that is necessary. If the output of the multiplier is to be fed to an integrator, then no output filtering is necessary at all. But if a high frequency response combined with good absolute accuracy are desired, the filtering problem becomes difficult. The following calculations show the approximate requirements put on the filter and the carrier wave forms.

Expressing a square wave in its fourier expansion:

$$(1)\quad Y = \frac{4}{\pi} E \left(\sin x + \frac{1}{3} \sin 3x + \cdots + \frac{1}{n} \sin nx \right) \quad n \text{ odd}$$

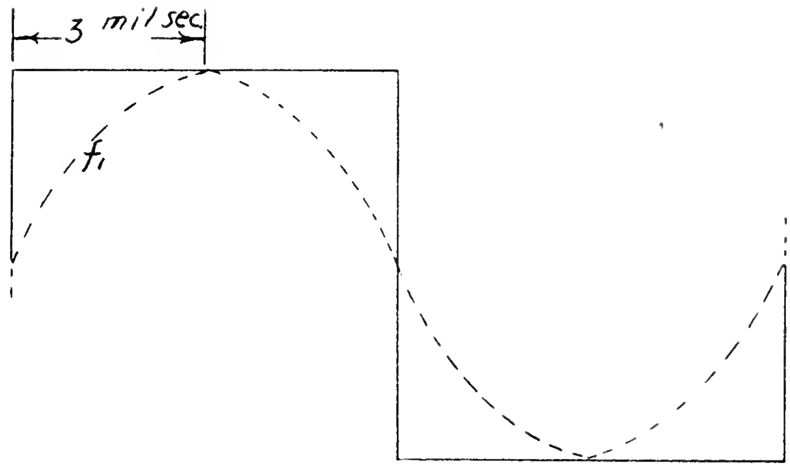
and realizing that the rise time is dependent upon the high frequency components of the wave form, consider the pass band requirements for one percent accuracy combined with a 3 millisecond response time. The function must be expanded until the accuracy is within one percent; or, in other words, the harmonics of the fundamental must be unattenuated until the accuracy of the resulting wave form is within one percent. The so-called fundamental frequency is determined from (Fig. 14) and the following:

$$(2)\quad \arc \sin 1 = \pi/2$$

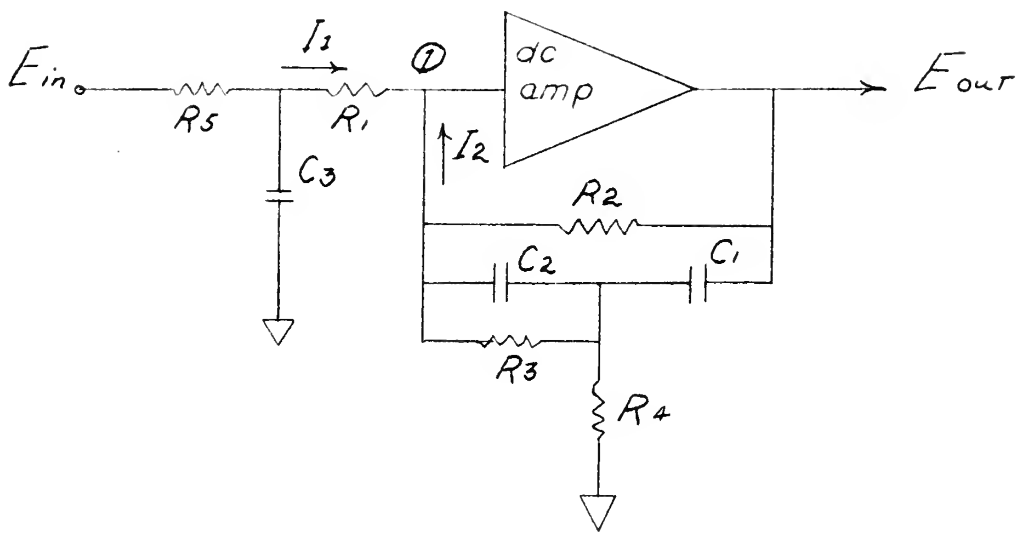
$$(3)\quad 2\pi f t = \pi/2$$

when $t = 3$ milliseconds

$f_1 = 83.4$ cycles per second



Filter bandpass requirements
Figure 14



Output filter network
Figure 15

Since consecutive harmonics add and subtract at the point of interest, and the series is convergent, the desired harmonic of the fundamental frequency can be obtained from equation (1) as follows:

$$(4) \qquad \Delta Y = \frac{4}{n\pi}$$

$$(5) \qquad .01 \times 2 = \frac{4}{n\pi} \qquad , n = 63$$

Therefore, the filter must pass the 63rd harmonic of 83.4 cycles, or 5.25 kc.

The second step in determining the necessary filter characteristics is to find the attenuation required at the carrier or switching repetition rate. In order to obtain a maximum product term of 50 volts in the form of XY/Z, the unfiltered carrier level must be 250 volts. One percent accuracy requires that only 0.5 volts of the carrier remain after filtering. Therefore, approximately 54 decibels attenuation at the fundamental of the carrier frequency will be required.

A low pass filter with an attenuation of 60 db/decade will be examined for the desired 3 millisecond response time with one percent error. Since 54 decibels attenuation is required, and assuming the attenuation to be zero to 5.25 kc, as determined above, the smallest carrier frequency allowable can be determined from the following:

$$(6) \quad f_{carrier} = f_{unattenuated} \left(\begin{smallmatrix}\text{attenuation}\\ \text{required}\end{smallmatrix}\right)\left(\text{filter}^{\prime}\ \text{attenuation}\right)$$

$$(7) \quad f_{carrier} = 5.25\ KC\,(54\,db)\left(\frac{10}{60\,db}\right)$$

$$f_{carrier} = 47.25\ K.C.$$

This means an average switching time of only 10.6 microseconds; 47.25 kc is almost 6.5 times the highest frequency with which one percent accuracy can be obtained in the present circuit.

Using a similar approach, it would be possible to attain an 8 millisecond response time with two percent accuracy with the existing 70 microsecond switching wave forms. This requires a 60 db/decade low pass filter with a flat response up to 900 cycles. A filter with these approximate characteristics can be designed as follows:

A low pass filter with a 60 db/decade attenuation can be obtained from the following network transfer function:

$$(8)\quad G(s) = \frac{(.1TS + 1)}{(TS + 1)(T^2 S^2 + .5TS + 1)}$$

where $T = \frac{1}{2\pi f t}$

"f" being the breakpoint for the composite attenuation versus frequency curve, or in the present case, 900 cycles per second,

$$T = 1.77 \times 10^{-4}$$

In order to make selection of components easier, let $T = 2 \times 10^{-4}$ thus making

$$(9)\quad G(s) = \frac{(2 \times 10^{-5} s + 1)}{(2 \times 10^{-4} S + 1)(4 \times 10^{-8} S^2 + 10^{-4} s + 1)}$$

The above transfer function can be obtained from the network shown in (Fig. 15).

In the circuit with the high gain d-c amplifier, the voltage at point 1 is effectively zero and $I_1 = -I_2$.

$$(10) \qquad \frac{E_{in}}{Z_1} = -\frac{E_o}{Z_2}$$

$$(11) \qquad \bar{Z}_1 = R_1(R_5 C_3 s + 1)$$

$$(12) \qquad \bar{Z}_2 = \frac{1}{R_2} + \frac{C_1 R_4 (R_3 C_2 s + 1) s}{(R_4 C_1 s + 1) R_3}$$

$$(13) \quad \bar{E}_o = -\bar{E}_{in} \times \frac{R_2}{R_1} \times \frac{(R_4 C_1 s + 1)}{(R_5 C_3 s + 1)\left[(R_2 R_4 C_1 C_2) s^2 + C_1 \frac{(R_4 + R_2 R_4)}{R_3} s + 1\right]}$$

Now choose values for the components of Z_1 and Z_2, to make the transfer function equivalent to the desired transfer function, keeping in mind that a gain of 10 is desired in the output stage. If R_2 is made 1 megohm, which is sufficiently large to reduce the effective load on the output amplifier, allowing a dynamic range of +50 to -50 volts, then $R_1 + R_5$ must equal 0.1 megohm to satisfy the gain of 10. It is also desirable to have as many components as possible with standard values and to make R_4 reasonably large, so as to keep the input impedance to the amplifier high. Utilizing

the above restrictions, the following identities can be used to find the desired component values:

(14) $$R_4 C_1 = 2 \times 10^{-5}$$

(15) $$R_5 C_3 = 2 \times 10^{-4}$$

(16) $$R_2 R_4 C_1 C_2 = 4 \times 10^{-8}$$

(17) $$R_5 = R_1 = 5 \times 10^{4}$$

The network then takes the following form:

$R_1 =$	50 K	ohms	$C_1 =$	1000	uuf
$R_2 =$	1 M	ohms	$C_2 =$	2000	uuf
$R_3 =$	250 K	ohms	$C_3 =$	4000	uuf
$R_4 =$	20 K	ohms			
$R_5 =$	50 K	ohms			

4. Tests and performance.

Both the accuracy and response time of the multiplying circuit were tested. Tests of the accuracy of the multiplying circuit were made by applying different values of constant voltages, ranging from -50 to +50 volts, to both the X and Y inputs with a constant -50 volts Z input. For precision testing, a zero reading micro ammeter was used, with a calibrated helipot across a known voltage source. Peak errors were found to be about one percent of the maximum output voltage and less than two percent absolute error at any output voltage. This accuracy was obtained when the multiplier was operated with a five kilocycle switching frequency and a simple high pass feed back network around the

final amplifier. The unstabilized d-c amplifiers were balanced about every ten minutes to attain this accuracy. In order to improve the response time to a step input to five milliseconds, the repetition rate was increased to 7.2 Kc, with a corresponding decrease in accuracy to two percent of the maximum output voltage.

The speed of response of the multiplying circuit was tested by applying a constant voltage to one input and a square wave to the other and viewing the output on an oscilloscope. A response time of eight milliseconds with two percent accuracy was attained. It is obvious that the response time can be decreased by increasing the switching frequency and adjusting the filter characteristics correspondingly, but this reduces the accuracy of the multiplier. Experimental results showed five percent accuracy with three millisecond response time. In order to improve the accuracy at a higher switching frequency, the transient time of the switching and carrier wave forms must be shortened. Suggestions for these improvements, without increasing the circuit complexity, are to select tubes for the square wave amplifiers with smaller input capacitance and higher transconductance tubes for the bistable multivibrator.

IV

CONCLUSION

The experimental results show that the time division multiplier circuit which was tested approaches the requirements set down for a universal analog multiplier. It is a four quadrant device with a relatively simple circuit. One percent accuracy can be obtained, but only at the expense of lowering the response time. Greater accuracy and faster speeds could be expected from the circuit after certain refinements. Automatically stabilized amplifiers must be used to eliminate the necessity of frequent balancing.

There is still a need for further development of simple, accurate, all electronic multipliers of low cost for use in computing operations on fast time scales.

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